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ON SOME CIRCULAR SUMMATION FORMULAS FOR THETA FUNCTIONS

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ABSTRACT. In his lost notebook, Ramanujan claimed that the "circular" summation of n^{th} power of his symmetric theta function $f(a,b)$ satisfies a factorization of the form $f(a,b)F(ab)$. In this paper, we obtain new circular summation formula of theta functions using the theory of elliptic functions. As an application, we also obtain few interesting identity of the theta functions.

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1. INTRODUCTION

In Ramanujan's notation [2, Ch.16, pp.34], we define the general theta function by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1}$$

For $q = e^{\pi i \tau}$ and $\mathrm{Im}(\tau) > 0$, the classical Jacobi's theta functions $\theta_i(z|\tau)$, $i = 1, 2, 3, 4$ are defined as follows:

$$\theta_1(z|\tau) = -iq^{1/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(m+1)} e^{(2m+1)iz}, \tag{2}$$

$$\theta_2(z|\tau) = q^{1/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(m+1)} e^{(2m+1)iz}, \tag{3}$$

$$\theta_3(z|\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} e^{2miz}, \tag{4}$$

$$\theta_4(z|\tau) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} e^{2miz}. \tag{5}$$

We get the following properties of $\theta_i(z|\tau), i = 1, 2, 3, 4$ employing (2)-(5):

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \quad \theta_1(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_1(z|\tau)\,, \tag{6}$$

$$\theta_2(z+\pi|\tau) = -\theta_2(z|\tau), \quad \theta_2(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_2(z|\tau)\,, \tag{7}$$

$$\theta_3(z+\pi|\tau) = \theta_3(z|\tau), \quad \theta_3(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_3(z|\tau)\,, \tag{8}$$

$$\theta_4(z+\pi|\tau) = \theta_4(z|\tau), \quad \theta_4(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_4(z|\tau)\,. \tag{9}$$

Applying induction on (6)-(9), we obtain

$$\theta_1(z+n\pi\tau|\tau) = (-1)^n q^{-n^2} e^{-2niz}\theta_1(z|\tau)\,, \tag{10}$$

$$\theta_2(z+n\pi\tau|\tau) = q^{-n^2} e^{-2niz}\theta_2(z|\tau)\,, \tag{11}$$

$$\theta_3(z+n\pi\tau|\tau) = q^{-n^2} e^{-2niz}\theta_3(z|\tau)\,, \tag{12}$$

$$\theta_4(z+n\pi\tau|\tau) = (-1)^n q^{-n^2} e^{-2niz}\theta_4(z|\tau)\,. \tag{13}$$

On page 54 of his lost notebook [14] (see also [1, pp.337]), Ramanujan recorded a statement which is now known as Ramanujan's circular summation:

Theorem 1.1. For any positive integer $n \geq 2$, if

$$U_r = a^{r(r+1)/2n} b^{r(r-1)/2n} \text{ and } V_r = a^{r(r-1)/2n} b^{r(r+1)/2n},$$

then

$$\sum_{r=0}^{n-1} U_r^n f^n \left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) = f(a,b)F_n(ab),$$

where

$$F_n(q) = 1 + 2nq^{(n-1)/2} + \cdots, n > 3.$$

Ramanujan's circular summation can be restated in terms of classical theta function $\theta_3(z|\tau)$ defined by (4).

Theorem 1.2. For any positive integer $n \geq 2$,

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n(z + k\pi\tau|n\tau) = F_n(\tau)\theta_3(z|\tau),$$

where

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots.$$

The proof of Theorem 1.1 was given by S. S. Rangachari in [15], by using Mumford's theory of theta functions [12] and few results on weight polynomials in coding theory. Later, Son [18] gave much elementary proof of Theorem 1.1. Recently, Xu [19] has given a very elementary and neat proof of the circular summation formula.

Applying the theory of elliptic functions, H. H. Chan, Z. G. Liu and S. T. Ng proved a dual form of Ramanujan's circular summation in. M. Boon, M. L. Glasser, J. Zak and I. J. Zucker [8] have proved an additive decomposition of θ_3. A general result that unifies the results of [8] and [9] is proved by Zeng [20]: For a, b, n and k any positive integers with $k = a + b$,

$$\sum_{s=0}^{kn-1} \theta_3^a\Big(\frac{z}{kn} + \frac{y}{a} + \frac{\pi s}{kn}\Big|\frac{\tau}{kn^2}\Big)\theta_3^b\Big(\frac{z}{kn} - \frac{y}{b} + \frac{\pi s}{kn}\Big|\frac{\tau}{kn^2}\Big) = \mathscr{C}_{33}\Big(a, b; \frac{y}{ab}, \frac{\tau}{kn^2}\Big)\theta_3(z|\tau),$$

where

$$\mathscr{C}_{33}(a,b;y,\tau) = kn \sum_{\substack{m_1,\cdots m_a, n_1, \cdots n_b=-\infty \\ m_1+\cdots+m_a+n_1+\cdots+n_b=0}}^{+\infty} q^{m_1^2+\cdots m_a^2+n_1^2+\cdots n_b^2} e^{2k(m_1+\cdots+m_a)iy}.$$

For more recent works on Ramanujan's circular summation one may refer [6], [7], [10], [13], [11], [16], [22], [23], [4], [9], [5] and [21].

Motivated by these works, in particular the works of Y. Cai, Y. Q. Bi and Q-M Luo [5], we obtain new Ramanujan's summation summation for four theta functions employing elliptic functions. In Section 2, we obtain our main results. In Section 3, we obtain interesting identities from the main results. Finally, we conclude the paper with some application.

2. Main Results

Theorem 2.1. Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have

$$\begin{aligned} &\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) \\ &\times\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = \mathscr{F}_{123}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\mathscr{F}_{123}(a,b,c;y,\tau) \\ &= lmni^a q^{\frac{a+b}{4}} \sum_{\substack{u_1,\cdots u_a, v_1,\cdots v_b, w_1, \cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_a} \\ &\qquad \times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b} \\ &\qquad \times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}. \end{aligned} \tag{15}$$

Theorem 2.2. Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have

$$\begin{aligned} &\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) \\ &\times\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = \mathscr{F}_{124}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \end{aligned} \tag{16}$$

where

$$\mathscr{F}_{124}(a,b,c;y,\tau)=lmni^a q^{\frac{a+b}{4}}$$
$$\times\sum_{\substack{u_1,\cdots u_a,v_1,\cdots,v_b,w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty}(-1)^{u_1+\cdots+u_a+w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b}$$
$$\times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}. \tag{17}$$

Theorem 2.3. Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have

$$\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\times\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)=\mathscr{F}_{134}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{18}$$

where

$$\mathscr{F}_{134}(a,b,c;y,\tau)=lmni^a q^{\frac{a}{4}}$$
$$\times\sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a=0}}^{+\infty}(-1)^{u_1+\cdots+u_a+w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+u_1+\cdots+u_a}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \tag{19}$$

Theorem 2.4. Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have

$$\sum_{s=0}^{lmn-1}\theta_2^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\times\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)=\mathscr{F}_{234}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{20}$$

where

$$\mathscr{F}_{234}(a,b,c;y,\tau)$$
$$=lmnq^{\frac{a}{4}}\sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a=0}}^{+\infty}(-1)^{w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+u_1+\cdots+u_a}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \tag{21}$$

Theorem 2.5. Let a, b, c, d, k, l, m and n are positive integers with $a+b+c+d=m$. If a, b, c, d are even, then we have

$$\sum_{s=0}^{klmn-1}\theta_1^a\Big(\frac{z}{klmn}+\frac{y}{a}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)\theta_2^b\Big(\frac{z}{klmn}+\frac{y}{b}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)$$
$$\theta_3^c\Big(\frac{z}{klmn}+\frac{y}{c}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)\theta_4^d\Big(\frac{z}{klmn}+\frac{y}{d}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)$$
$$=\mathscr{F}_{1234}\Big(a,b,c,d;\frac{y}{abcd},\frac{\tau}{klmn^2}\Big)\theta_3(z|\tau), \tag{22}$$

where

$$\mathscr{F}_{1234}(a,b,c,d;y,\tau)=klmni^aq^{\frac{a+b}{4}}$$
$$\times\sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c,x_1,\cdots x_d=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots w_c+x_1+\cdots+x_d)\\ +a+b=0}}^{+\infty}(-1)^{u_1+\cdots+u_a+x_1+\cdots+x_d}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+x_1^2+\cdots+x_d^2+u_1+\cdots+u_a+v_1+\cdots+v_b}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \tag{23}$$

Proof: Let $f(z)$ be the left side of (14). Then

$$f(z+\pi)=$$
$$\sum_{s=1}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)+\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}\Big|\frac{\tau}{lmn^2}\Big). \tag{24}$$

$$f(z+\pi)=$$
$$\sum_{s=1}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)+(-1)^{a+b}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}\Big|\frac{\tau}{lmn^2}\Big). \tag{25}$$

Since a and b are even, we have from (24) and (25)

$$f(z+\pi)=f(z). \tag{26}$$

Again from (24) and a, b, c even, we have

$$f(z+\pi\tau) = \sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)\times$$
$$\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)$$
$$= q^{-1}e^{-2iz}\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\times\,\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = q^{-1}e^{-2iz}f(z). \tag{27}$$

By (26) and (27), we have constructed an elliptic function $f(z)/\theta_3(z|\tau)$ with double periods π and $\pi\tau$ and only have a simple pole at $z=\pi/2+\pi\tau/2$ in the period parallelogram. Hence the function $f(z)/\theta_3(z|\tau)$ is a constant, say $F_{123}(a,b,c;y,\tau)$, i.e.

$$\frac{f(z)}{\theta_3(z|\tau)} = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau),$$

or, equivalently

$$f(z) = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau).$$

Hence

$$\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau). \tag{28}$$

Employing (2), (3) and (5), we obtain

$$F_{123}(a,b,c;y,\tau)\sum_{m=-\infty}^{\infty} q^{m^2}e^{2miz}$$
$$= (-1)^a i^a q^{\frac{a+b}{4lmn^2}} \sum_{s=0}^{lmn-1} \sum_{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots,w_c=-\infty}^{\infty} (-1)^{u_1+\cdots+u_c}$$
$$\times\, q^{\frac{u_1^2+\cdots+u_a^2+v_1^2+\cdots+v_a^2+w_1^2+\cdots+w_a^2+u_1+\cdots+u_a+v_1+\cdots+v_b}{lnm^2}}$$
$$\times\, e^{\frac{\{2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b\}}{lmn}iz}$$
$$\times\, e^{\left\{\frac{2(u_1+\cdots+u_a)+a}{a}+\frac{2(v_1+\cdots+v_a)+b}{b}+\frac{w_1+\cdots+w_a}{c}\right\}iy}$$
$$\times\, e^{\frac{\{2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b\}}{lmn}i\pi s}. \tag{29}$$

The constant term on both sides of (29), we have

$$
\begin{aligned}
&F_{123}(a,b,c;y,\tau)\\
&= lmni^a q^{\frac{a+b}{4lmn^2}} \sum_{\substack{u_1,\cdots u_a, v_1,\cdots v_b, w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+\\ w_1+\cdots w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_c}\\
&\qquad \times q^{\frac{u_1^2+\cdots+u_a^2+v_1^2+\cdots+v_a^2+w_1^2+\cdots+w_a^2+u_1+\cdots+u_a+v_1+\cdots+v_b}{lnm^2}}\\
&\qquad \times e^{\frac{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_a)+ab(w_1+\cdots+w_a)+abc\}}{abc}iy}.
\end{aligned}
$$

It is clear that

$$
F_{123}(a,b,c;y,\tau) = \mathscr{F}_{123}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big),
$$

where

$$
\begin{aligned}
&\mathscr{F}_{123}(a,b,c;y,\tau)\\
&= lmni^a q^{\frac{a+b}{4}} \sum_{\substack{u_1,\cdots u_a, v_1,\cdots v_b, w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_a}\\
&\qquad \times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b}\\
&\qquad \times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}.
\end{aligned}
$$

This complete the proof of Theorem 2.1.
Proof of Theorem 2.2 − 2.5 are similar.

3. Interesting Identities of Main Results

L. C. Shen [17] has obtained the Fourier series expansion of triple product of Jacobi theta functions. These expansion can be converted into difference of theta functions as follows:

Lemma 3.1. From (2), (3), (4) and (5), we have

$$
\begin{aligned}
&\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)\\
&= -iq^{3/2}(q^2;q^2)_\infty^2\{e^{41z}\theta_4(3z+2\pi\tau|3\tau) - e^{-4iz}\theta_4(3z-2\pi\tau|3\tau)\}, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
&\theta_1(z|\tau)\theta_2(z|\tau)\theta_4(z|\tau)\\
&= iq^{3/2}(q^2;q^2)_\infty^2\{e^{41z}\theta_3(3z+2\pi\tau|3\tau) - e^{-4iz}\theta_3(3z-2\pi\tau|3\tau)\}, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&\theta_1(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau)\\
&= iq^{5/4}(q^2;q^2)_\infty^2\{e^{21z}\theta_2(3z+\pi\tau|3\tau) - e^{-2iz}\theta_2(3z-\pi\tau|3\tau)\}, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau)\\
&= -iq^{1/2}(q^2;q^2)_\infty^2\{e^{21z}\theta_1(3z+\pi\tau|3\tau) - e^{-2iz}\theta_1(3z-\pi\tau|3\tau)\}. \qquad (33)
\end{aligned}
$$

Proof: We have from [17, Proposition 2.1]

$$\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)$$

$$= 2q^{3/2}(q^2;q^2)_\infty^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+4n} \sin(6n+4)z$$

$$= -iq^{3/2}(q^2;q^2)_\infty^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+4n} \Big(e^{(6n+4)iz} - e^{-(6n+4)iz}\Big)$$

$$= -iq^{3/2}(q^2;q^2)_\infty^2 \Big(e^{4iz} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} e^{2ni(2\pi\tau+3z)}$$

$$-e^{-4iz} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} e^{2ni(2\pi\tau-3z)}\Big). \tag{34}$$

We obtain (30) by replacing n by $-n$ in second summation and by the definition of $\theta_4(z|\tau)$ (5). Similarly, (31) and (33) can be proved. Now

$$\theta_1(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau)$$

$$= 2q^{1/4}(q^2;q^2)_\infty^2 \sum_{n=-\infty}^{\infty} q^{3n^2+4n} \sin(6n+1)z$$

$$= iq^{1/4}(q^2;q^2)_\infty^2 \Big(\sum_{n=-\infty}^{\infty} q^{3n^2+n} e^{-(6n+1)iz} - \sum_{n=-\infty}^{\infty} q^{3n^2+n} e^{(6n+1)iz}\Big)$$

$$= iq^{5/4}(q^2;q^2)_\infty^2 \Big(e^{2iz} \sum_{n=-\infty}^{\infty} q^{3n^2-3n} e^{i(2n-1)(3z+\pi\tau)}$$

$$-e^{-2iz} \sum_{n=-\infty}^{\infty} q^{3n^2-3n} e^{i(2n-1)(3z-\pi\tau)}\Big). \tag{35}$$

(32) is achieved by replacing n by $-n$ in the first summation and n by $n-1$ in the second summation and employing (3).

Theorems $2.1 - 2.4$ are closely related with the Lemma 3.1.

Theorem 3.2. Setting $a=b=c=t$ in (14) and employing (30), we obtain

$$\sum_{s=0}^{lmn-1} \left(\begin{matrix} e^{4i(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})}\theta_4\Big(3(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})+\frac{2\pi\tau}{lmn^2}|\frac{3\tau}{lmn^2}\Big) \\ -e^{-4i(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})}\theta_4\Big(3(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})-\frac{2\pi\tau}{lmn^2}|\frac{3\tau}{lmn^2}\Big) \end{matrix} \right)^t$$

$$= \mathscr{K}_{123}\Big(t,t,t;\frac{y}{t^3},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{36}$$

where

$$\mathscr{K}_{123}(t,t,t;y,\tau)$$
$$= \frac{lmn}{q^t(q^2;q^2)_\infty^{2t}} \sum_{\substack{u_1,\cdots u_t,v_1,\cdots v_t,w_1,\cdots w_t=-\infty \\ u_1+\cdots+u_t+v_1+\cdots+v_t+w_1+\cdots+w_t+t=0}}^{+\infty} (-1)^{u_1+\cdots+u_t+w_1+\cdots+w_t}$$
$$\times q^{u_1^2+\cdots+u_t^2+v_1^2+\cdots+v_t^2+w_1^2+\cdots w_t^2+u_1+\cdots+u_t+v_1+\cdots+v_t}$$
$$\times e^{2\{t^2(u_1+\cdots+u_t+v_1+\cdots+v_t+w_1+\cdots+w_t)+t\}iy}. \tag{37}$$

On setting $a=b=c=t$ in identities in (16), (18) and (20) and employing identities (31), (32) and (33) respectively, we obtain similar results.

4. Applications of Main Results

In this Section, we gives some special cases of Theorems 2.1 − 2.5 and obtain some interesting identities of theta functions.

Corollary 4.1. For l and n positive even integers, we have

$$\sum_{s=0}^{6ln-1} \theta_1^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)$$
$$\theta_3^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big) = 6ln\theta_1^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau) \tag{38}$$

Proof: Setting $a=b=c=2$ in (14)

$$\mathscr{F}_{123}(2,2,2;y,\tau) = -6lnq$$
$$\sum_{\substack{u_1,u_2,v_1,v_2,w_1,w_2=-\infty \\ u_1+u_2+v_1+v_2+w_1+w_2+2=0}}^{+\infty} (-1)^{u_1+u_a} q^{u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2+u_1+u_2+v_1+v_2}$$
$$\times e^{8\{(u_1+u_1)+(v_1+v_2)+(w_1+w_2)+2\}iy}$$
$$= -6lnq \sum_{u_1,u_2,v_1,v_2,w_1,w_2=-\infty}^{+\infty} (-1)^{u_1+u_a} q^{u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2+u_1+u_2+v_1+v_2}$$
$$= 6ln\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau). \tag{39}$$

Changing τ by $\frac{\tau}{6ln^2}$ in (39), we obtain Corollary 4.1.

Setting $l=n=1$ in Corollary 4.1,

$$\sum_{s=0}^{5} \theta_1^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)\theta_2^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)\theta_3^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)$$
$$= 6\theta_1^2\Big(0\Big|\frac{\tau}{6}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6}\Big)\theta_3^2\Big(0\Big|\frac{\tau}{6}\Big)\theta_3(z|\tau). \tag{40}$$

Setting $z \longmapsto 6z$, $y \longmapsto 2y$ and $\tau \longmapsto 6\tau$ in (40) and simplifying, we obtain

$$\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)+\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_4^2(z+y|\tau)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)$$
(41)
$$=6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_3(6z|6\tau).$$

Setting $y=0$ in (41), we obtain

$$\theta_1^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)+\theta_1^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)$$
$$\theta_3^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)=\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big).$$

More similar identities can be obtain from Theorem $2.2-2.5$.

Remark: The Jacobi transformation formulas for theta functions on applying Jacobi imaginary transformation formulas are given by

$$\begin{aligned}
\theta_1\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= -i\sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_1(z|\tau)\,,\\
\theta_2\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_4(z|\tau)\,,\\
\theta_3\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_3(z|\tau)\,,\\
\theta_4\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_2(z|\tau)\,.
\end{aligned}$$

Using the above identities, we can obtain imaginary transformation formulas of our Main results.

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